IS THE LOCATION OF THE SUPREMUM OF A STATIONARY PROCESS NEARLY UNIFORMLY DISTRIBUTED?

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Abstract. It is, perhaps, surprising that the location of the unique supremum of a stationary process on an interval can fail to be uniformly distributed over that interval. We show that this distribution is absolutely continuous in the interior of the interval and describe very specific conditions the density has to satisfy. We establish universal upper bounds on the density and demonstrate their optimality.

1. Introduction

The structure of the excursion sets of stochastic processes and random fields over different levels has attracted plenty of interest in the last several years; much of the recent progress is described in the recent book Adler and Taylor (2007). A particular effort went to understanding the so-called persistent topology of the excursion sets, which addresses, roughly, the changes in the structure of the excursion sets as the level changes; see e.g. Adler et al. (2010). Such changes depend strongly on the locations of both global and local maxima of the random field in the domain.

The present work arises from an obvious attempt to undersand the effect of stationarity of the process, or random field, on such questions. This, clearly, requires imposing assumptions on the time domain of the random field and, in this paper, we look at the simplest possible case: that of stationary stochastic processes in continuous, one-dimensional, time, and we will consider the location of its global supremum over a compact interval.

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It turns out that answering even this, apparently simple question, leads to unexpected insights.

We now discuss out setup more formally. Let $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ be a stationary process. If the sample paths of the process are upper semi-continuous, then the process is bounded from above on any compact interval $[0,T]$, and its supremum over that interval is attained. We are interested in the location of that supremum within the interval $[0,T]$.

It is, of course, entirely possible that the supremum of the process in the interval $[0,T]$ is not unique (i.e. that it is achieved at more than one point). In that case one could be more specific and take, for example, the left-most point in which the largest value over the interval is achieved, as the location of the supremum. In this paper we will sometimes deal with the situation in which, on an event of probability 1, the supremum is achieved at a single point. In either case it is easy to check that the location of the supremum is a well defined random variable.

The stationarity of the process seems to guarantee that the location of the supremum is uniformly distributed over the interval, or does it? Of course, if the supremum is not uniquely attained, and we choose to work with its left-most position, then this choice can, perhaps, skew the distribution of the location to the left, but one can be forgiven for believing, even for a moment, that in the case of a uniquely attained supremum, it has to be uniformly located. However, already the examples in Section 9.4 of Leadbetter et al. (1983) show that even in the case of Gaussian processes, the supremum can be located, with a positive probability, at one of the endpoints of the interval and, furthermore, the remaining mass in the interior of the interval does not have to be uniformly distributed there.

It is, of course, the endpoints of the interval that are responsible for the lack of uniformity. In a sense, the points near the ends of the interval have "fewer local competitors" for being the supremum than the points further from the endpoints do. But exactly how far from having the uniform distribution can the location of the supremum be? In this paper we give a very

detailed answer to this question by showing that this distribution is absolutely continuous in the interior of the interval and describing very specific conditions its density must satisfy. This is done in Section 2. Our results turn out to be quite complete. In fact, we show in a companion paper Samorodnitsky and Shen (2011) that, for a very broad class of stationary processes with a uniquely achieved supremum, our description actually gives all possible distributions of its location. In the present paper we start with treating a general upper semi-continuous stationary process and (with one exception) allowing the process to have multiple supremum locations within an interval. We proceed with establishing extra conditions the density has to satisfy if the process satisfies certain assumptions. In Section 4 we provide the sharpest possible universal upper bounds on the density both in the general case and in the case of time-reversible stationary processes.

2. Notation and assumptions on the stationary process

For the remainder of this paper $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ is a stationary process with upper semi-continuous sample paths, defined on some probability space $\big(\Omega, \mathcal{F}, P\big)$. For a compact interval $[a,b]$, we will denote by

$$\tau_{\mathbf{X},[a,b]} = \min\big\{t \in [a,b] :\ X(t) = \sup_{a\leq s\leq b} X(s)\big\}.$$

That is, $\tau_{\mathbf{X},[a,b]}$ is the first time the overall supremum in the interval $[a,b]$ is achieved. It is elementary to check that $\tau_{\mathbf{X},([a,b])}$ is a well defined random variable. If $a=0$, we will use the single variable notation $\tau_{\mathbf{X},b}$.

We denote by $F_{\mathbf{X},[a,b]}$ the law of $\tau_{\mathbf{X},[a,b]}$; it is a probability measure on the interval $[a,b]$. If $a=0$, we have the corresponding single variable notation $F_{\mathbf{X},b}$. The following statements are obvious.

Lemma 2.1. (i) *For any $\Delta \in \mathbb{R}$,*

$$F_{\mathbf{X},[\Delta,T+\Delta]}(\cdot) = F_{\mathbf{X},T}(\cdot - \Delta)\,.$$

(ii) *For any intervals $[c,d] \subseteq [a,b]$,*

$$F_{\mathbf{X},[a,b]}(B) \leq F_{\mathbf{X},[c,d]}(B) \text{ for any Borel set } B \subset [c,d].$$

The discussion of the leftmost supremum location $\tau_{\mathbf{X},[a,b]}$ in the sequel applies equally well to the rightmost supremum location, for instance, by considering the time-reversed stationary process $(X(-t),\, t \in \mathbb{R})$. In some cases we will find it convenient to assume that the supremum is achieved at a unique location. Formally, for $T > 0$ we denote by $X_*(T) = \sup_{0 \le t \le T} X(t)$ the largest value of the process in the interval $[0,T]$, and consider the set

$$\Omega_T = \bigl\{\omega \in \Omega :\ X(t_i) = X_*(T),\ i = 1,2, \text{ for two different } t_1, t_2 \in [0,T]\bigr\}.$$

It is easy to see that Ω_T is a measurable set. The following assumption says that, on a set of probability 1, the supremum over interval $[0,T]$ is uniquely achieved.

Assumption U_T: $P(\Omega_T) = 0$.

In our previous notation, under Assumption U_T, $\tau_{\mathbf{X},[a,b]}$ is the unique point at which the supremum over the interval $[0,T]$ is achieved, and $F_{\mathbf{X},T}$ is the law of that point.

Even though many of our results do not require it, the most complete description of the distribution of the location of the supremum that we have requires the following, additional, assumption.

Assumption L:

$$K := \lim_{\varepsilon \downarrow 0} \frac{P\bigl(\mathbf{X} \text{ has a local maximum in } (0,\varepsilon)\bigr)}{\varepsilon} < \infty \,.$$

It is easy to check that the limit in Assumption L exists. If, for example, the process $\mathbf{X}$ has differentiable sample paths, then a sufficient condition for Assumption L is that the expected number of times the process $Y(t) = X'(t)$, $t \in \mathbb{R}$ crosses zero in a unit time interval is finite; the latter can be checked using, for instance, Theorem 7.2.4 in Leadbetter et al. (1983).

Assumption L rules out existence of "too frequent" local extrema of the sample paths. For sample continuous processes this also rules out rapid oscillation of the sample paths possessed, for instance, by the Gaussian Ornstein-Uhlenbeck process of Example 3.5 below. In fact, we will presently see that, at least for sample continuous processes, under Assumption L the process has, with probability 1, sample paths of locally bounded variation.

Lemma 2.2. *Let* $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ *be a stationary sample upper semi-continuous process satisfying Assumption L. Then, for any* $T > 0$*, on an event of probability 1 the process has finitely many local maxima and minima in the interval* $(0, T)$*. In particular, if the process is sample continuous, then its sample paths are, on event of probability 1, of locally bounded variation.*

Proof. For notational simplicity we take $T = 1$. For $n = 1, 2, \ldots$ let

$$N_n = \sum_{i=1}^{2^n} \mathbf{1}\left(\text{a point in } \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \text{ is a local maximum of } \mathbf{X}\right).$$

Clearly, the sequence N_n is nondecreasing, and $N_n \to N_\infty$, where N_∞ is the total number of local maxima of $\mathbf{X}$ in the interval $[0, 1)$. By the monotone convergence theorem,

$$EN_\infty = \lim_{n\to\infty} EN_n$$

$$\leq \limsup_{n\to\infty} 2^n P\big(\mathbf{X} \text{ has a local maximum in } (0, 2^{-n})\big) \leq K.$$

Therefore, $N_\infty < \infty$ a.s. Since between any two distinct local minima there is a local maximum, the number of local minima in $[0, 1)$ is a.s. finite as well. Since a sample continuous process must have a monotone path between any two consecutive local extrema, the lemma has been proved. □

3. Description of the possible distributions of the location of the supremum

We start with a result showing existence of a density in the interior of the interval $[0, T]$ of the leftmost location of the supremum in that interval for any upper semi-continuous stationary process, as well as conditions this density has to satisfy. Only one of the statements of the theorem requires Assumption U_T, in which case the statement applies to the unique location of the supremum. See Remark 3.2 in the sequel.

Theorem 3.1. *Let* $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ *be a stationary sample upper semi-continuous process. Then the restriction of the law* $F_{\mathbf{X},T}$ *to the interior* $(0, T)$ *of the interval is absolutely continuous. The density, denoted by* $f_{\mathbf{X},T}$*, can be taken to be equal to the right derivative of the cdf* $F_{\mathbf{X},T}$*, which exists at every*

point in the interval $(0,T)$. *In this case the density is right continuous, has left limits, and has the following properties.*

(a) The limits

$$f_{\mathbf{X},T}(0+) = \lim_{t\to 0} f_{\mathbf{X},T}(t) \text{ and } f_{\mathbf{X},T}(T-) = \lim_{t\to T} f_{\mathbf{X},T}(t)$$

exist.

(b) The density has a universal upper bound given by

$$f_{\mathbf{X},T}(t) \le \max\left(\frac{1}{t}, \frac{1}{T-t}\right), \ 0 < t < T\,. \tag{3.1}$$

(c) Assume that the process satisfies Assumption U_T. *Then the density is bounded away from zero:*

$$\inf_{0<t<T} f_{\mathbf{X},T}(t) > 0\,. \tag{3.2}$$

(d) The density has a bounded variation away from the endpoints of the interval. Furthermore, for every $0 < t_1 < t_2 < T$,

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) \le \min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big) + \min\big(f_{\mathbf{X},T}(t_2),\ f_{\mathbf{X},T}(t_2-)\big)\,, \tag{3.3}$$

where

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) = \sup \sum_{i=1}^{n-1} \big|f_{\mathbf{X},T}(s_{i+1}) - f_{\mathbf{X},T}(s_i)\big|$$

is the total variation of $f_{\mathbf{X},T}$ *on the interval* (t_1,t_2), *and the supremum is taken over all choices of* $t_1 < s_1 < \ldots < s_n < t_2$.

(e) The density has a bounded positive variation at the left endpoint and a bounded negative variation at the right endpoint. Furthermore, for every $0 < \varepsilon < T$,

$$TV^{+}_{(0,\varepsilon)}(f_{\mathbf{X},T}) \le \min\big(f_{\mathbf{X},T}(\varepsilon), f_{\mathbf{X},T}(\varepsilon-)\big) \tag{3.4}$$

and

$$TV^{-}_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) \le \min\big(f_{\mathbf{X},T}(T-\varepsilon), f_{\mathbf{X},T}(T-\varepsilon-)\big)\,, \tag{3.5}$$

where for any interval $0 \le a < b \le T$,

$$TV^{\pm}_{(a,b)}(f_{\mathbf{X},T}) = \sup \sum_{i=1}^{n-1} \big(f_{\mathbf{X},T}(s_{i+1}) - f_{\mathbf{X},T}(s_i)\big)_{\pm}$$

is the positive (negative) variation of $f_{\mathbf{X},T}$ *on the interval* (a,b), *and the supremum is taken over all choices of* $a < s_1 < \ldots < s_n < b$.

(f) The limit $f_{\mathbf{X},T}(0+) < \infty$ *if and only if* $TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) < \infty$ *for some (equivalently, any)* $0 < \varepsilon < T$*, in which case*

$$TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) \le f_{\mathbf{X},T}(0+) + \min\bigl(f_{\mathbf{X},T}(\varepsilon), f_{\mathbf{X},T}(\varepsilon-)\bigr)\,. \tag{3.6}$$

Similarly, $f_{\mathbf{X},T}(T-) < \infty$ *if and only if* $TV_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) < \infty$ *for some (equivalently, any)* $0 < \varepsilon < T$*, in which case*

$$TV_{(T-\varepsilon,T)}(f_{\mathbf{X},T}) \le \min\bigl(f_{\mathbf{X},T}(T-\varepsilon), f_{\mathbf{X},T}(T-\varepsilon-)\bigr) + f_{\mathbf{X},T}(T-)\,. \tag{3.7}$$

Proof. Choose $0 < \delta < T/2$. We claim that for every $\delta \le t \le T-\delta$, for every $\rho > 0$ and every $0 < \varepsilon < \delta\rho/(1+\rho)$

$$P\bigl(t < \tau_{\mathbf{X},T} \le t+\varepsilon\bigr) \le \varepsilon(1+\rho)\max\left(\frac{1}{t}, \frac{1}{T-t}\right)\,. \tag{3.8}$$

This statement, once proved, will imply absolute continuity of $F_{\mathbf{X},T}$ on the interval $(\delta, T-\delta)$ and, since $\delta > 0$ can be taken to be arbitrarily small, also on $(0,T)$. Further, (3.8) will imply that the version of the density given by

$$f_{\mathbf{X},T}(t) = \limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} P\bigl(t < \tau_{\mathbf{X},T} \le t+\varepsilon\bigr),\ 0 < t < T\,,$$

satisfies the bound (3.1).

We proceed to prove (3.8). Suppose that, to the contrary, (3.8) fails for some $\delta \le t \le T-\delta$ and $0 < \varepsilon < \delta\rho/(1+\rho)$. Choose

$$\varepsilon < \theta < \frac{\rho}{1+\rho}\delta$$

and $0 < a < t < b < T$ such that

$$\min\bigl(t, T-t\bigr) - \theta < b-a < \min\bigl(t, T-t\bigr) - \varepsilon\,.$$

For $a \le s \le b$, by stationarity, we have

$$P\bigl(s < \tau_{\mathbf{X},[s-t,s-t+T]} \le s+\varepsilon\bigr) > \varepsilon(1+\rho)\max\left(\frac{1}{t}, \frac{1}{T-t}\right)\,. \tag{3.9}$$

Further, let $a \le s_1 < s_1+\varepsilon \le s_2 \le b$. We check next that

$$\bigl\{s_j < \tau_{\mathbf{X},[s_j-t,s_j-t+T]} \le s_j+\varepsilon,\ j=1,2\bigr\} = \emptyset\,. \tag{3.10}$$

Indeed, let Ω_{s_1,s_2} be the event in (3.10). Note that the intervals $(s_1, s_1+\varepsilon)$ and $(s_2, s_2+\varepsilon)$ are disjoint and, by the choice of the parameters a and b, each of these two intervals is a subinterval of both $[s_1-t, s_1-t+T]$ and $[s_2-t, s_2-t+T]$. Therefore, on the event Ω_{s_1,s_2} we cannot have

$$X\bigl(\tau_{\mathbf{X},[s_1-t,s_1-t+T]}\bigr) < X\bigl(\tau_{\mathbf{X},[s_2-t,s_2-t+T]}\bigr)\,,$$

for otherwise $\tau_{\mathbf{X},[s_1-t,s_1-t+T]}$ would fail to be a location of the maximum over the interval $[s_1-t,s_1-t+T]$. For the same reason on the event Ω_{s_1,s_2} we cannot have

$$X\big(\tau_{\mathbf{X},[s_1-t,s_1-t+T]}\big) > X\big(\tau_{\mathbf{X},[s_2-t,s_2-t+T]}\big)\,.$$

Finally, on the event Ω_{s_1,s_2} we cannot have

$$X\big(\tau_{\mathbf{X},[s_1-t,s_1-t+T]}\big) = X\big(\tau_{\mathbf{X},[s_2-t,s_2-t+T]}\big)\,,$$

for otherwise $\tau_{\mathbf{X},[s_2-t,s_2-t+T]}$ would fail to be the leftmost location of the maximum over the interval $[s_2-t,s_2-t+T]$. This establishes (3.10).

We now apply (3.9) and (3.10) to the points $s_i = a+i\varepsilon,\ i=0,1,\ldots,\lceil (b-a)/\varepsilon\rceil - 1$. We have

$$1 \geq P\left(\bigcup_{i=0}^{\lceil (b-a)/\varepsilon\rceil-1}\big\{s_i < \tau_{\mathbf{X},[s_i-t,s_i-t+T]} \leq s_i+\varepsilon\big\}\right)$$

$$= \sum_{i=0}^{\lceil (b-a)/\varepsilon\rceil-1} P\big(s_i < \tau_{\mathbf{X},[s_i-t,s_i-t+T]} \leq s_i+\varepsilon\big) > \frac{b-a}{\varepsilon}\varepsilon(1+\rho)\max\left(\frac{1}{t},\frac{1}{T-t}\right)$$

$$> \Big(\min\big(t,T-t\big)-\theta\Big)(1+\rho)\max\left(\frac{1}{t},\frac{1}{T-t}\right)$$

$$> \left(1-\frac{\delta}{\min(t,T-t)}\frac{\rho}{1+\rho}\right)(1+\rho) \geq \left(1-\frac{\rho}{1+\rho}\right)(1+\rho) = 1$$

by the choice of θ. This contradiction proves (3.8).

Before proceeding with the proof of Theorem 3.1, we pause to prove the following important lemma.

Lemma 3.1. *Let $0 \leq \Delta < T$. Then for every $0 \leq \delta \leq \Delta$, $f_{\mathbf{X},T-\Delta}(t) \geq f_{\mathbf{X},T}(t+\delta)$ almost everywhere in $(0,T-\Delta)$. Furthermore, for every such δ and every $\varepsilon_1,\varepsilon_2 \geq 0$, such that $\varepsilon_1+\varepsilon_2 < T-\Delta$,*

$$\int_{\varepsilon_1}^{T-\Delta-\varepsilon_2}\big(f_{\mathbf{X},T-\Delta}(t) - f_{\mathbf{X},T}(t+\delta)\big)\,dt \tag{3.11}$$

$$\leq \int_{\varepsilon_1}^{\varepsilon_1+\delta} f_{\mathbf{X},T}(t)\,dt + \int_{T-\Delta-\varepsilon_2+\delta}^{T-\varepsilon_2} f_{\mathbf{X},T}(t)\,dt\,.$$

Proof. We simply use Lemma 2.1. For any Borel set $B \subseteq (0,T-\Delta)$ we have

$$\int_B f_{\mathbf{X},T-\Delta}(t)\,dt = P\big(\tau_{\mathbf{X},T-\Delta} \in B\big) \geq P\big(\tau_{\mathbf{X},[-\delta,T-\delta]} \in B\big)$$

$$= \int_B f_{\mathbf{X},[-\delta,T-\delta]}(t)\,dt = \int_B f_{\mathbf{X},T}(t+\delta)\,dt\,,$$

which shows that $f_{\mathbf{X},T-\Delta}(t) \geq f_{\mathbf{X},T}(t+\delta)$ almost everywhere in $(0,T-\Delta)$.

For (3.11), notice that by Lemma 2.1,

$$\int_{\varepsilon_1}^{T-\Delta-\varepsilon_2} \big(f_{\mathbf{X},T-\Delta}(t) - f_{\mathbf{X},T}(t+\delta)\big)\,dt$$

$$\begin{aligned}
&= P\big(\tau_{\mathbf{X},T-\Delta} \in (\varepsilon_1, T-\Delta-\varepsilon_2)\big) - P\big(\tau_{\mathbf{X},T} \in (\varepsilon_1+\delta, T-\Delta-\varepsilon_2+\delta)\big)\\
&= P\big(\tau_{\mathbf{X},T} \notin (\varepsilon_1+\delta, T-\Delta-\varepsilon_2+\delta)\big) - P\big(\tau_{\mathbf{X},T-\Delta} \notin (\varepsilon_1, T-\Delta-\varepsilon_2)\big)\\
&= P\big(\tau_{\mathbf{X},T} \in [0,\varepsilon_1+\delta)\big) + P\big(\tau_{\mathbf{X},T} \in (T-\Delta-\varepsilon_2+\delta, T]\big)\\
&\quad - P\big(\tau_{\mathbf{X},T-\Delta} \in [0,\varepsilon_1)\big) - P\big(\tau_{\mathbf{X},T-\Delta} \in (T-\Delta-\varepsilon_2, T-\Delta]\big)\\
&= P\big(\tau_{\mathbf{X},T} \in (\varepsilon_1,\varepsilon_1+\delta)\big) + \Big(P\big(\tau_{\mathbf{X},T} \in [0,\varepsilon_1)\big) - P\big(\tau_{\mathbf{X},T-\Delta} \in [0,\varepsilon_1)\big)\Big)\\
&\quad + P\big(\tau_{\mathbf{X},T} \in (T-\Delta-\varepsilon_2+\delta, T-\varepsilon_2)\big)\\
&\quad + \Big(P\big(\tau_{\mathbf{X},T} \in (T-\varepsilon_2, T]\big) - P\big(\tau_{\mathbf{X},[\Delta,T]} \in (T-\varepsilon_2, T]\big)\Big)\\
&\leq P\big(\tau_{\mathbf{X},T} \in (\varepsilon_1,\varepsilon_1+\delta)\big) + P\big(\tau_{\mathbf{X},T} \in (T-\Delta-\varepsilon_2+\delta, T-\varepsilon_2)\big)\\
&= \int_{\varepsilon_1}^{\varepsilon_1+\delta} f_{\mathbf{X},T}(t)\,dt + \int_{T-\Delta-\varepsilon_2+\delta}^{T-\varepsilon_2} f_{\mathbf{X},T}(t)\,dt\,,
\end{aligned}$$

as required. □

We return now to the proof of Theorem 3.1. Our next goal is to prove that the cdf $F_{\mathbf{X},T}$ is right differentiable at every point in the interval $(0,T)$. Since we already know that $F_{\mathbf{X},T}$ is absolutely continuous on $(0,T)$, the set

(3.12) $$A = \big\{t \in (0,T):\ F_{\mathbf{X},T} \text{ is not right differentiable at } t\big\}$$

has Lebesgue measure zero. Define next

(3.13)
$$B = \big\{t \in A^c:\ f_{\mathbf{X},T} \text{ restricted to } A^c \text{ does not have a right limit at } t\big\}.$$

We claim that the set B is at most countable. To see this, we define for $t \in A^c$

$$L(t) = \limsup_{s\downarrow t,\, s\in A^c} f_{\mathbf{X},T}(s), \quad l(t) = \liminf_{s\downarrow t,\, s\in A^c} f_{\mathbf{X},T}(s)\,.$$

Our claim about set B will follow once we check that for any $0 < \varepsilon < T/2$ and $\theta > 0$, the set

$$B_{\varepsilon,\theta} = \big\{t \in A^c \cap (\varepsilon, T-\varepsilon):\ L(t) - l(t) > \theta\big\}$$

is finite. In fact, we will show that the cardinality of $B_{\varepsilon,\theta}$ cannot be larger than $4/(\varepsilon\theta)$. If not, let $N > 4/(\varepsilon\theta)$ and find points $\varepsilon < t_1 < t_2 < \ldots < t_N < T-\varepsilon$. Choose $\delta > 0$ so small that $\delta < \varepsilon/2$ and

$$0 < \delta < \frac{1}{2}\min\Big(t_1-\varepsilon, t_2-t_1, \ldots, t_N - t_{N_1}, T-\varepsilon-t_N\Big)\,.$$

Let now $i = 1, \ldots, N$ and choose a sequence $s_n \downarrow t_i$, $s_n \in A^c$, such that $f_{\mathbf{X},T}(s_n) \to L(t_i)$. Consider n so large that $s_n - t_i < \delta/3$, and let

$$j \geq \frac{3}{\delta - (s_n - t_i)}$$

be an integer. We have

$$P\big(\tau_{\mathbf{X},T-\delta} \in (t_i - \delta, t_i)\big) \geq \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor - 1} P\big(\tau_{\mathbf{X},T-\delta} \in (t_i-(k+1)/j, t_i - k/j)\big)\,,$$

and for each k as in the sum,

$$h_k := s_n - t_i + \frac{k+1}{j} \in (0,\delta]\,.$$

Therefore, by Lemma 2.1

$$\begin{aligned} &P\big(\tau_{\mathbf{X},T-\delta} \in (t_i-\delta, t_i)\big) \\ &\geq \sum_{k=0}^{\lfloor j(\delta-(s_n-t_i))\rfloor - 1} P\big(\tau_{\mathbf{X},T} \in (t_i - (k+1)/j + h_k, t_i - k/j + h_k)\big) \\ &= \lfloor j(\delta-(s_n-t_i))\rfloor P\big(\tau_{\mathbf{X},T} \in (s_n, s_n+1/j)\big) \to (\delta - (s_n-t_i)) f_{\mathbf{X},T}(s_n) \end{aligned}$$

as $j \to \infty$. Letting $n \to \infty$, we conclude that

$$P\big(\tau_{\mathbf{X},T-\delta} \in (t_i-\delta, t_i)\big) \geq \delta L(t_i),\ i = 1, \ldots, N\,. \tag{3.14}$$

Similarly, for $i = 1, \ldots, N$ choose a sequence $w_n \downarrow t_i$, $w_n \in A^c$, such that $f_{\mathbf{X},T}(w_n) \to l(t_i)$. For large n and j we have

$$\begin{aligned} &P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, t_i+\delta)\big) = P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, w_n)\big) + P\big(\tau_{\mathbf{X},T+\delta} \in (w_n, w_n+\delta)\big) \\ &\leq P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, w_n)\big) + \sum_{k=0}^{\lceil \delta j\rceil - 1} P\big(\tau_{\mathbf{X},T+\delta} \in (w_n + k/j, w_n + (k+1)/j)\big)\,. \end{aligned}$$

For each k as in the sum above,

$$h_k := \frac{k}{j} \in [0,\delta]\,.$$

Therefore, by Lemma 2.1,

$$P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, t_i+\delta)\big)$$

$$\le P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, w_n)\big) + \lceil \delta j \rceil P\big(\tau_{\mathbf{X},T} \in (w_n, w_n + 1/j)\big).$$

Letting, once again, first $j \to \infty$ and then $n \to \infty$, we conclude that

$$P\big(\tau_{\mathbf{X},T+\delta} \in (t_i, t_i + \delta)\big) \le \delta l(t_i),\ i = 1, \ldots, N\,. \tag{3.15}$$

Now we use the estimate in Lemma 3.1 as follows. By the definition of the point t_i and the smallness of δ,

$$N\delta\theta \le P\left(\tau_{\mathbf{X},T-\delta} \in \bigcup_{i=1}^{N} (t_i - \delta, t_i)\right) - P\left(\tau_{\mathbf{X},T+\delta} \in \bigcup_{i=1}^{N} (t_i, t_i + \delta)\right)$$
$$= \int_{\cup_{i=1}^{N}(t_i-\delta,t_i)} \big(f_{\mathbf{X},T-\delta}(t) - f_{\mathbf{X},T+\delta}(t+\delta)\big)\,.$$

Using the fact that

$$\bigcup_{i=1}^{N} (t_i - \delta, t_i) \subset (\varepsilon - \delta, T - \varepsilon)\,,$$

and that, by Lemma 3.1, the integrand above is a.e. nonnegative, we have by the estimate in that lemma that the integral above does not exceed

$$\int_{\varepsilon-\delta}^{T-\varepsilon} \big(f_{\mathbf{X},T-\delta}(t) - f_{\mathbf{X},T+\delta}(t+\delta)\big)\,dt$$
$$\le \int_{\varepsilon-\delta}^{\varepsilon} f_{\mathbf{X},T+\delta}(t)\,dt + \int_{T-\varepsilon+\delta}^{T-\varepsilon+2\delta} f_{\mathbf{X},T+\delta}(t)\,dt\,.$$

Applying the already proved (3.1), we conclude that

$$N\delta\theta \le 2\frac{\delta}{\varepsilon - \delta} \le \frac{4\delta}{\varepsilon}\,,$$

and this contradicts the assumption that we can choose $N > 4/(\varepsilon\theta)$. This proves that the set B in (3.13) is at most countable. We notice, further, that

$$f_{\mathbf{X},T}(t) = \lim_{s \downarrow t} \frac{1}{s - t} P\big(t < \tau_{\mathbf{X},T} \le s\big) \tag{3.16}$$
$$= \lim_{s \downarrow t} \frac{1}{s - t} \int_t^s f_{\mathbf{X},T}(w)\,dw \;=\; \lim_{w \downarrow t,\, w \in A^c \setminus B} f_{\mathbf{X},T}(w)$$

for every $t \in A^c \setminus B$ (recall the set A is defined in (3.12)).

Now we are ready to prove that the right derivative of the cdf $F_{\mathbf{X},T}$ exists at every point in the interval $(0,T)$. Suppose, to the contrary, that this is not so. Then there is $t \in (0,T)$ and real numbers $a < b$ such that

$$\liminf_{\varepsilon \downarrow 0} \frac{F_{\mathbf{X},T}(t+\varepsilon) - F_{\mathbf{X},T}(t)}{\varepsilon} < a < b < \limsup_{\varepsilon \downarrow 0} \frac{F_{\mathbf{X},T}(t+\varepsilon) - F_{\mathbf{X},T}(t)}{\varepsilon}.$$

This implies that there is a sequence $t_n \downarrow t$ with $t_n \in A^c \setminus B$ for each n such that

$$f_{\mathbf{X},T}(t_{2n-1}) > b, \ f_{\mathbf{X},T}(t_{2n}) < a \text{ for all } n = 1, 2, \ldots.$$

We can and will choose t_1 so close to t that $t_1 < (T+t)/2$.

Notice that by (3.16), for every $n = 1, 2, \ldots$ there is $\delta_n > 0$ such that

$$f_{\mathbf{X},T}(w) > b \text{ a.e. in } (t_{2n-1}, t_{2n-1} + \delta_{2n-1}),$$

$$f_{\mathbf{X},T}(w) < a \text{ a.e. in } (t_{2n}, t_{2n} + \delta_{2n})$$

for $n = 1, 2, \ldots$.

Let now $m \geq 1$, and consider $s > 0$ so small that both $s < \min_{n=1,\ldots,2m} \delta_n$ and $t_1 < (T+t)/2 - s$. Observe that

$$\int_t^{(T+t)/2} \big(f_{\mathbf{X},T}(w+s) - f_{\mathbf{X},T}(w)\big)_+ \, dw$$

$$\geq \int_t^{t+s} \sum_{i=0}^{\lfloor (T-t)/2s \rfloor - 1} \big(f_{\mathbf{X},T}(w+(i+1)s) - f_{\mathbf{X},T}(w+is)\big)_+ \, dw\,,$$

and for every point $w \in (t, t+s)$, each one of the intervals $(t_n, t_n + \delta_n)$, $n = 1, \ldots, 2m$, contains at least one of the points in the finite sequence $w + is$, $i = 0, 1, \ldots, \lfloor (T-t)/2s \rfloor - 1$. By construction, apart from a set of points $w \in (t, t+s)$ of measure zero, those points of the kind $w + is$ that fall in the odd-numbered intervals satisfy $f_{\mathbf{X},T}(w+is) > b$, and those points that fall in the even-numbered intervals satisfy $f_{\mathbf{X},T}(w+is) < a$. We conclude that

$$\sum_{i=0}^{\lfloor (T-t)/2s \rfloor - 1} \big(f_{\mathbf{X},T}(w+(i+1)s) - f_{\mathbf{X},T}(w+is)\big)_+ \geq m(b-a)$$

a.e. in $(t, t+s)$. Therefore, for all $s > 0$ small enough,

$$\int_t^{(T+t)/2} \big(f_{\mathbf{X},T}(w+s) - f_{\mathbf{X},T}(w)\big)_+ \, dw \geq sm(b-a)$$

and, since m can be taken arbitrarily large, we conclude that

$$\lim_{s \downarrow 0} \frac{1}{s} \int_t^{(T+t)/2} \big(f_{\mathbf{X},T}(w+s) - f_{\mathbf{X},T}(w)\big)_+ \, dw = \infty\,. \tag{3.17}$$

We will see that this is, however, impossible, and the resulting contradiction will prove that the right derivative of the cdf $F_{\mathbf{X},T}$ exists at every point in the interval $(0, T)$.

Indeed, recall that by Lemma 3.1, for all $s > 0$ small enough,

$$f_{\mathbf{X},T-2s}(w-s) \geq f_{\mathbf{X},T}(w+s) \text{ a.e. on } (s, T-s) \supset (t, (T+t)/2).$$

Therefore, for such s,

$$\begin{aligned}
&\int_t^{(T+t)/2} \big(f_{\mathbf{X},T}(w+s) - f_{\mathbf{X},T}(w)\big)_+ \, dw \\
&\leq \int_t^{(T+t)/2} \big(f_{\mathbf{X},T-2s}(w-s) - f_{\mathbf{X},T}(w)\big)_+ \, dw \\
&\leq \int_{t-s}^{(T+t)/2-s} \big(f_{\mathbf{X},T-2s}(w) - f_{\mathbf{X},T}(w+s)\big) \, dw
\end{aligned}$$

since, by another application of Lemma 3.1, the integrand is a.e. nonnegative over the range of integration. Applying (3.11), we see that

$$\begin{aligned}
&\int_t^{(T+t)/2} \big(f_{\mathbf{X},T}(w+s) - f_{\mathbf{X},T}(w)\big)_+ \, dw \\
&\leq \int_{t-s}^{t} f_{\mathbf{X},T}(w) \, dw + \int_{(T+t)/2}^{(T+t)/2+s} f_{\mathbf{X},T}(w) \, dw \, .
\end{aligned}$$

However, we already know that the density $f_{\mathbf{X},T}$ is bounded on any subinterval of $(0,T)$ that is bounded away from both endpoints. Therefore, the upper bound obtained above shows that (3.17) is impossible. Hence the existence of the right derivative everywhere, which then coincides with the version of the density $f_{\mathbf{X},T}$ chosen above.

Next we check that this version of the density is right continuous. To this end we recall that we already know that the set A in (3.12) is empty. Next, we rule out existence of a point $t \in (0,T)$ such the limit of $f_{\mathbf{X},T}(s)$ as $s \downarrow t$ over $s \in B^c$ does not exist. Suppose that, to the contrary, that such t exists. This means that there are real numbers $a < b$ and a sequence $t_n \downarrow t$ with $t_n \in B^c$ for each n such that

$$f_{\mathbf{X},T}(t_{2n-1}) > b, \ f_{\mathbf{X},T}(t_{2n}) < a \text{ for all } n = 1, 2, \ldots .$$

However, we have already established that such a sequence cannot exist.

As in (3.16), we see that for every $t \in (0,T)$

$$f_{\mathbf{X},T}(t) = \lim_{s \downarrow t, \, s \in B^c} f_{\mathbf{X},T}(s)$$

and, since the set B is at most countable, the restriction to $s \in B^c$ in the above limit statement can be removed. This proves right continuity of the

version of the density density given by the right derivative of $F_{\mathbf{X},T}$. The proof of existence of left limits is similar.

Next, we address the variation of the version of the density we are working with away from the endpoints of the interval $(0,T)$. Let $0<t_1<t_2<T$. We start with a preliminary calculation. Let $0<r_n<T-t_2$. Introduce the notation

$$\begin{aligned} C_+ &= \bigl\{t\in(t_1,t_2):\, f_{\mathbf{X},T}(t+r_n)\geq f_{\mathbf{X},T}(t)\bigr\}\,,\\ C_- &= \bigl\{t\in(t_1,t_2):\, f_{\mathbf{X},T}(t+r_n)< f_{\mathbf{X},T}(t)\bigr\}\,, \end{aligned}$$

so that

$$\int_{t_1}^{t_2}\bigl|f_{\mathbf{X},T}(t+r_n)-f_{\mathbf{X},T}(t)\bigr|\,dt$$
$$=\int_{C_+}\bigl(f_{\mathbf{X},T}(t+r_n)-f_{\mathbf{X},T}(t)\bigr)\,dt+\int_{C_-}\bigl(f_{\mathbf{X},T}(t)-f_{\mathbf{X},T}(t+r_n)\bigr)\,dt\,.$$

To estimate the two terms we will once again use Lemma 3.1. Since

$$f_{\mathbf{X},T-r_n}(t)\geq f_{\mathbf{X},T}(r_n+t)\ \text{a.e. on}\ (0,T-r_n)\supset(t_1,t_2)$$

for n large enough, for such n, we have the upper bound

$$\begin{aligned} \int_{C_+}\bigl(f_{\mathbf{X},T}(t+r_n)-f_{\mathbf{X},T}(t)\bigr)\,dt &\leq \int_{C_+}\bigl(f_{\mathbf{X},T-r_n}(t)-f_{\mathbf{X},T}(t)\bigr)\,dt\\ &\leq \int_{t_1}^{t_2}\bigl(f_{\mathbf{X},T-r_n}(t)-f_{\mathbf{X},T}(t)\bigr)\,dt\,. \end{aligned}$$

We now once again use (3.11) to conclude that for all n large, we have

$$\int_{C_+}\bigl(f_{\mathbf{X},T}(t+r_n)-f_{\mathbf{X},T}(t)\bigr)\,dt\leq\int_{t_2}^{t_2+r_n}f_{\mathbf{X},T}(t)\,dt$$

so that

$$\limsup_{n\to\infty}\frac{1}{r_n}\int_{C_+}\bigl(f_{\mathbf{X},T}(t+r_n)-f_{\mathbf{X},T}(t)\bigr)\,dt\leq f_{\mathbf{X},T}(t_2)\,.$$

Similarly, by Lemma 3.1,

$$f_{\mathbf{X},T}(t+r_n)\geq f_{\mathbf{X},T+r_n}(t+r_n)\ \text{a.e. on}\ (0,T-r_n)\supset(t_1,t_2)$$

for n large enough, and we obtain, for such n, using (3.11)

$$\begin{aligned} \int_{C_-}\bigl(f_{\mathbf{X},T}(t)-f_{\mathbf{X},T}(t+r_n)\bigr)\,dt &\leq \int_{C_-}\bigl(f_{\mathbf{X},T}(t)-f_{\mathbf{X},T+r_n}(t+r_n)\bigr)\,dt\\ &\leq \int_{t_1}^{t_2}\bigl(f_{\mathbf{X},T}(t)-f_{\mathbf{X},T+r_n}(t+r_n)\bigr)\,dt\leq\int_{t_1}^{t_1+r_n}f_{\mathbf{X},T+r_n}(t)\,dt\,. \end{aligned}$$

This can, in turn, be bounded from above both by

$$\int_{t_1}^{t_1+r_n} f_{\mathbf{X},T}(t)\,dt$$

and by

$$\int_{t_1}^{t_1+r_n} f_{\mathbf{X},T}(t-r_n)\,dt = \int_{t_1-r_n}^{t_1} f_{\mathbf{X},T}(t)\,dt\,.$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{r_n}\int_{C_-} \big(f_{\mathbf{X},T}(t) - f_{\mathbf{X},T}(t+r_n)\big)\,dt \le \min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big)\,.$$

Overall, we have proved that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{r_n}\int_{t_1}^{t_2} \big|f_{\mathbf{X},T}(t+r_n) - f_{\mathbf{X},T}(t)\big|\,dt \\ \le \min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big) + f_{\mathbf{X},T}(t_2)\,. \end{aligned} \tag{3.18}$$

To relate (3.18) to the total variation of the density $f_{\mathbf{X},T}$ over the interval (t_1,t_2), we notice first that by the right continuity of the density, it is enough to consider the regularly spaced points $s_i = t_1 + ir_n,\ i = 1,\dots,n$, where $r_n = (t_2-t_1)/(n+1)$ for some $n = 1,2,\dots$. Write

$$\int_{t_1}^{t_2} \big|f_{\mathbf{X},T}(t+r_n) - f_{\mathbf{X},T}(t)\big|\,dt = \int_{t_1}^{t_1+r_n} \sum_{i=0}^{n} \big|f_{\mathbf{X},T}(t+(i+1)r_n) - f_{\mathbf{X},T}(t+ir_n)\big|\,dt$$

and observe that

$$\lim_{n\to\infty} \sum_{i=0}^{n} \big|f_{\mathbf{X},T}(t+(i+1)r_n) - f_{\mathbf{X},T}(t+ir_n)\big| \ge TV_{(t_1,t_2)}(f_{\mathbf{X},T})$$

uniformly in $t \in (t_1,t_2)$. Therefore, by (3.18)

$$\min\big(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\big) + f_{\mathbf{X},T}(t_2) \ge \limsup_{n\to\infty} \frac{1}{r_n}\int_{t_1}^{t_2} \big|f_{\mathbf{X},T}(t+r_n) - f_{\mathbf{X},T}(t)\big|\,dt$$

$$\ge \limsup_{n\to\infty} \frac{1}{r_n}\int_{t_1}^{t_1+r_n} \sum_{i=0}^{n} \big|f_{\mathbf{X},T}(t+(i+1)r_n) - f_{\mathbf{X},T}(t+ir_n)\big|\,dt \ge TV_{(t_1,t_2)}(f_{\mathbf{X},T})\,.$$

Now the bound (3.3) follows from the obvious fact that

$$TV_{(t_1,t_2)}(f_{\mathbf{X},T}) = \lim_{\varepsilon\downarrow 0} TV_{(t_1,t_2-\varepsilon)}(f_{\mathbf{X},T}),\,.$$

Furthermore, the proof of (3.4) and (3.5) is the same as the proof of (3.3), with each one using one side of the two-sided calculation performed above for (3.3).

Next, the boundedness of the positive variation of the density at zero, clearly, implies that the limit $f_{\mathbf{X},T}(0+) = \lim_{t\downarrow 0} f_{\mathbf{X},T}(t)$ exists, while the

boundedness of the negative variation of the density at T implies that the limit $f_{\mathbf{X},T}(T-) = \lim_{t\uparrow T} f_{\mathbf{X},T}(t)$ exists as well. If $TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) < \infty$ for some $0 < \varepsilon < T$, then, trivially, $f_{\mathbf{X},T}(0+) < \infty$. On the other hand, if $f_{\mathbf{X},T}(0+) < \infty$, then the same argument as we used in proving (3.3), shows that for any $0 < \varepsilon < T$,

$$TV^-_{(0,\varepsilon)}(f_{\mathbf{X},T}) \leq f_{\mathbf{X},T}(0+)\,,$$

which, together with (3.4), both shows that $TV_{(0,\varepsilon)}(f_{\mathbf{X},T}) < \infty$ and proves (3.6). One can prove the statement of part (f) of the theorem concerning the behaviour of the density at the right end point of the interval in the same way.

It only remains to prove part (c) of the theorem, namely the fact that the version of the density given by the right derivative of the cdf $F_{\mathbf{X},T}$ is bounded away from zero. Recall that Assumption U_T is in effect here.

Suppose, to the contrary, that (3.2) fails and introduce the notation

$$t_1 = \inf\bigl\{s \in (0,T) : \inf_{0<t<s} f_{\mathbf{X},T}(t) = 0\bigr\},$$

$$t_2 = \sup\bigl\{s \in (0,T) : \inf_{s<t<T} f_{\mathbf{X},T}(t) = 0\bigr\}\,.$$

Clearly, $0 \leq t_1 \leq t_2 \leq T$. We claim that,

$$\text{if } t_1 < t_2, \text{ then } f_{\mathbf{X},T}(t) = 0 \text{ for all } t_1 < t < t_2. \tag{3.19}$$

We start with the case $0 < t_1 < t_2 < T$. Notice that, in this case,

$$\min\bigl(f_{\mathbf{X},T}(t_1), f_{\mathbf{X},T}(t_1-)\bigr) = \min\bigl(f_{\mathbf{X},T}(t_2), f_{\mathbf{X},T}(t_2-)\bigr) = 0\,.$$

By (3.3) the density is constant on the interval (t_1, t_2). If $f_{\mathbf{X},T}(t_1) = 0$ then, by the right continuity of the density the constant must be equal to zero, so (3.19) is immediate. If $f_{\mathbf{X},T}(t_1-) = 0$ then, given $\varepsilon > 0$, choose $0 < s < t_1$ such that $f_{\mathbf{X},T}(s) \leq \varepsilon$. By (3.3) we know that $TV_{(s,t_2)}(f_{\mathbf{X},T}) \leq \varepsilon$, which implies that $f(t) \leq 2\varepsilon$ on (s, t_2), hence also on (t_1, t_2). Letting $\varepsilon \to 0$ proves (3.19). If either $t_1 = 0$ and/or $t_2 = T$, then (3.19) can be proved using a similar argument and the continuity of the density at 0 and at T shown in part (a) of the theorem. Furthermore, we also have

$$\text{if } t_1 = t_2, \text{ then } \min\bigl(f_{\mathbf{X},T}(t_1),\ f_{\mathbf{X},T}(t_1-)\bigr) = 0\,, \tag{3.20}$$

with the obvious conventions in the case $t_1 = t_2$ coincide with one of the endpoints of the interval.

It follows from (3.19), (3.20) and Lemma 3.1 that for any $\Delta > 0$,

$$f_{\mathbf{X},T+\Delta}(t) = 0 \text{ for } t_1 < t < t_2 + \Delta. \tag{3.21}$$

Furthermore, we know by Lemma 2.1 that

$$F_{\mathbf{X},T+\Delta}([0,t_1]) \le F_{\mathbf{X},T}([0,t_1]) \tag{3.22}$$

and

$$F_{\mathbf{X},T+\Delta}([t_2+\Delta, T+\Delta]) \le F_{\mathbf{X},T}([t_2,T])\,. \tag{3.23}$$

Note that for $\Delta > 0$ all the quantities in the above equations refer to the leftmost location $\tau_{\mathbf{X},T+\Delta}$ of the supremum, which is no longer assumed to be unique.

Since the distributions $F_{\mathbf{X},T}$ and $F_{\mathbf{X},T+\Delta}$ have equal total masses (equal to one), it follows from (3.21), (3.22) and (3.23) that the latter two inequalities must hold as equalities for all relevant sets. We concentrate on the resulting equation

$$F_{\mathbf{X},T+\Delta}([t_2+\Delta, T+\Delta]) = F_{\mathbf{X},T}([t_2,T])\,. \tag{3.24}$$

Since we are working with the leftmost supremum location on a larger interval, we can write for $\Delta > 0$

$$P\big(\tau_{\mathbf{X},T} \in [t_2,T]\big) = P\big(\tau_{\mathbf{X},[-\Delta,T]} \in [t_2,T]\big)$$
$$+P\big(\tau_{\mathbf{X},T} \in [t_2,T],\, \tau_{\mathbf{X},[-\Delta,T]} \in [-\Delta,0)\big).$$

Using Lemma 2.1 and (3.24) we see that

$$P\big(\tau_{\mathbf{X},T} \in [t_2,T],\, \tau_{\mathbf{X},[-\Delta,T]} \in [-\Delta,0)\big) = 0\,,$$

which implies that, if $\Delta > T - t_2$, then

$$P\big(\tau_{\mathbf{X},T} \in [t_2,T],\, \sup_{-\Delta\le t\le -\Delta+T-t_2} X(t) \ge \sup_{t_2\le t\le T} X(t)\big) = 0\,. \tag{3.25}$$

Pick $\delta > T$. Using (3.25) with $\Delta = n\delta - t_2$, $n = 1,2,\ldots$, we see that

$$Y_n < Y_0 \text{ a.e. on } \{\tau_{\mathbf{X},T} \in [t_2,T]\} \text{ for } n = 1,2,\ldots,$$

where $Y_n = \sup_{t_2-n\delta\le t\le T-n\delta} X(t)$, $n = 0,1,2,\ldots$. Note, however, that the sequence $(Y_n,\, n = 0,1,2,\ldots)$ is stationary, and for a stationary sequence it

is impossible that, on a set of positive probability, $Y_0 > Y_n$ for $n = 1, 2, \ldots$ (this is clear for an ergodic sequence; in general one can use the ergodic decomposition). We conclude that

$$P\big(\tau_{\mathbf{X},T} \in [t_2, T]\big) = 0\,. \tag{3.26}$$

Reversing the direction of time (or, equivalently, switching to the rightmost supremum location on a larger interval) and using Assumption U_T, we also have

$$P\big(\tau_{\mathbf{X},T} \in [0, t_1]\big) = 0\,. \tag{3.27}$$

However, (3.19), (3.26) and (3.27) rule out any possible mass of the distribution $F_{\mathbf{X},T}$. This contradiction shows that, under Assumption U_T, the version of the density given by the right derivative of the cdf $F_{\mathbf{X},T}$ is bounded away from zero. This completes the proof of the theorem. □

Remark 3.2. The following example shows that the statement of part (c) of Theorem 3.1 may fail without Assumption U_T.

Let $(x(t),\, t \in \mathbb{R})$ be a continuous periodic function with period 1, for which $t = 0$ is a global maximum. Let U be a standard uniform random variable. Then $(X(t) = x(t+U),\, t \in \mathbb{R})$ is a continuous stationary process, that always attains its global maximum in the interval $[0, 1]$. Therefore, with $T > 1$, we have $f_{\mathbf{X},T}(t) = 0$ for $1 \le t < T$.

Next we describe what extra restrictions on the distribution of the location of the supremum, in addition to the statements of Theorem 3.1, Assumption L of Section 2 imposes. Again, one of the statements of the theorem requires Assumption U_T. See Remark 3.4 for a discussion.

Theorem 3.3. *Let $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ be a stationary sample upper semi-continuous process, satisfying Assumption L. Then the version of the density $f_{\mathbf{X},T}$ of the leftmost location of the supremum in the interval $[0, T]$ described in Theorem 3.1 has the following additional properties.*

(a) $f_{\mathbf{X},T}(0+) < \infty$, $f_{\mathbf{X},T}(T-) < \infty$ and $TV_{(0,T)}(f_{\mathbf{X},T}) \le f_{\mathbf{X},T}(0+) + f_{\mathbf{X},T}(T-)$. In particular, the density has a bounded variation on the entire interval $(0, T)$.

(b) Assume additionally that the process is sample continuous and satisfies Assumption U_T. *Then either* $f_{\mathbf{X},T}(t) = 1/T$ *for all* $0 < t < T$, *or* $\int_0^T f_{\mathbf{X},T}(t)\,dt < 1$.

Note that part (b) of Theorem 3.3 says that, unless the location of the supremum is uniformly distributed in the interval $(0, T)$, the supremum is achieved, with a positive probability, at an endpoint of the interval.

It turns out that the description of the possible densities of the location of the supremum under Assumption U_T, given in Theorem 3.3, is complete, in the sense that for any function f satisfying the constraints described in the theorem, there is a sample continuous stationary satisfying Assumption U_T and Assumption L, for which f is the density of the supremum location. This is shown in Samorodnitsky and Shen (2011).

Proof of Theorem 3.3. Assumption L and stationarity imply that for any $0 < t < T$,

$$f_{\mathbf{X},T}(t) = \lim_{\varepsilon\downarrow 0} \frac{P(\tau_{\mathbf{X},T} \in (t, t+\varepsilon))}{\varepsilon}$$

$$\leq \limsup_{\varepsilon\downarrow 0} \frac{P(\mathbf{X} \text{ has a local maximum in } (t, t+\varepsilon))}{\varepsilon}$$

$$= \limsup_{\varepsilon\downarrow 0} \frac{P(\mathbf{X} \text{ has a local maximum in } (0, \varepsilon))}{\varepsilon} \leq K\,.$$

This proves finiteness of $f_{\mathbf{X},T}(0+) < \infty$ and $f_{\mathbf{X},T}(T-)$. The rest of the statement in part (a) follows from (3.6) by letting $\varepsilon \uparrow T$.

We now prove part (b). Assume that $P(\tau_{\mathbf{X},T} = 0 \text{ or } T) = 0$. By stationarity this implies that $\tau_{\mathbf{X},[T,2T]} \in (T, 2T)$ with probability 1. We first prove that

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) \neq X\big(\tau_{\mathbf{X},T}\big)\Big) = 0\,. \tag{3.28}$$

By symmetry, it is enough to prove the one-sided claim

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) < X\big(\tau_{\mathbf{X},T}\big)\Big) = 0\,. \tag{3.29}$$

Indeed, suppose, to the contrary, that the probability in (3.29) is positive. Under Assumption U_T we can use the continuity from below of measures to

see that there is $\varepsilon > 0$ such that

$$p := P\Big(X\big(\tau_{\mathbf{X},T}\big) > X\big(\tau_{\mathbf{X},[T,2T]}\big) + \varepsilon,\ X\big(\tau_{\mathbf{X},T}\big) > \max_{t \in L_T,\, t \neq \tau_{\mathbf{X},T}} X(t) + \varepsilon\Big) > 0\,.$$

Here L_T is the (a.s. finite) set of the local maxima of $\mathbf{X}$ in the interval $(0,T)$.

Next, by the uniform continuity of the process $\mathbf{X}$ on $[0,T]$, there is $n \geq 1$ such that

$$P\Big(\sup_{0 \leq s < t \leq T,\, t-s \leq T/n} \big|X(t) - X(s)\big| > \varepsilon/2\Big) \leq p/2\,.$$

We immediately conclude by the law of total probability that there is $i = 1,\dots,n$ such that $P(A_i) > 0$, where

$$A_i = \Big\{X\big(\tau_{\mathbf{X},T}\big) > X\big(\tau_{\mathbf{X},[T,2T]}\big) + \varepsilon,\ X\big(\tau_{\mathbf{X},T}\big) > \max_{t \in L_T,\, t \neq \tau_{\mathbf{X},T}} X(t) + \varepsilon,$$

$$(i-1)T/n < \tau_{\mathbf{X},T} < iT/n,\ \sup_{(i-1)T/n \leq s,t \leq iT/n} \big|X(t) - X(s)\big| \leq \varepsilon/2\Big\}.$$

However, on the event A_i, $X(iT/n) = \sup_{iT/n \leq t \leq 2T} X(t)$, implying that $\tau_{\mathbf{X},[iT/n,iT/n+T]} = iT/n$. By stationarity, this contradicts the assumption $P(\tau_{\mathbf{X},T} = 0) = 0$. This contradiction proves (3.29) and, hence, also (3.28).

Next, we check that

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) = X\big(\tau_{\mathbf{X},T}\big),\ \tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} < T\Big) = 0\,. \tag{3.30}$$

Indeed, suppose that, to the contrary, the probability above is positive. By the continuity from below of measures, there is $\varepsilon > 0$ such that

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) = X\big(\tau_{\mathbf{X},T}\big),\ \tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} < T - \varepsilon\Big) > 0\,.$$

Take $n > 2T/\varepsilon$. By the law of total probability there are $i_1, i_2 = 1,\dots,n$ such that $P(A_{i_1,i_2}) > 0$, where

$$A_{i_1,i_2} = \Big\{X\big(\tau_{\mathbf{X},[T,2T]}\big) = X\big(\tau_{\mathbf{X},T}\big),\ \tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} < T - \varepsilon,$$

$$(i_1-1)T/n < \tau_{\mathbf{X},T} < i_1T/n,\ T + (i_2-1)T/n < \tau_{\mathbf{X},[T,2T]} < T + i_2T/n\Big\}.$$

By the choice of n, $T + i_2T/n - (i_1-1)T/n < T$, so that, on the event A_{i_1,i_2}, the process $\mathbf{X}$ has at least two points, $\tau_{\mathbf{X},T}$ and $\tau_{\mathbf{X},[T,2T]}$, at which the supremum over the interval $[(i_1-1)T/n,\ (i_1-1)T/n + T]$ is achieved. By stationarity, this contradicts Assumption U_T. This contradiction proves (3.30).

Finally, we check that

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) = X\big(\tau_{\mathbf{X},T}\big),\ \tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} > T\Big) = 0\,. \tag{3.31}$$

The proof is similar to the proof of (3.29), so we only sketch the argument. Suppose that, to the contrary, the probability in (3.31) is positive. Use the continuity of measures to see that the probability remains positive if we require that $\tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} > T + \varepsilon$ for some $\varepsilon > 0$. Next, use Assumption U_T to separate the value of $X\big(\tau_{\mathbf{X},T}\big)$ from the values of $\mathbf{X}$ at other local maxima in $(0,T)$ and, finally, use the uniform continuity of the process $\mathbf{X}$ to show that there is a point $T < b < 2T$ and an event of positive probability on which $\tau_{\mathbf{X},[b-T,b]} = b$. By stationarity, this contradicts the assumption $P(\tau_{\mathbf{X},T} = T) = 0$.

Combining (3.28), (3.30) and (3.31), we see that the assumption $P(\tau_{\mathbf{X},T} = 0 \text{ or } T) = 0$ implies that

$$P\Big(X\big(\tau_{\mathbf{X},[T,2T]}\big) = X\big(\tau_{\mathbf{X},T}\big),\ \tau_{\mathbf{X},[T,2T]} - \tau_{\mathbf{X},T} = T\Big) = 1 \tag{3.32}$$

Let $0 < a < b < T$. We have by stationarity,

$$P\big(\tau_{\mathbf{X},T} \in (0, b-a)\big) = P\big(\tau_{\mathbf{X},[a,a+T]} \in (a,b)\big)$$

$$= P\big(\tau_{\mathbf{X},[a,a+T]} \in (a,b),\ \tau_{\mathbf{X},T} \in (0,a)\big) + P\big(\tau_{\mathbf{X},[a,a+T]} \in (a,b),\ \tau_{\mathbf{X},T} \in (a,T)\big)\,.$$

By (3.32), if $\tau_{\mathbf{X},T} \in (0,a)$, then $\tau_{\mathbf{X},[T,2T]} \in (T, T+a)$ and $X\big(\tau_{\mathbf{X},[T,2T]}\big) > \sup_{t\in[a,b]} X(t)$. Therefore, the first term in the right hand side above vanishes. Similarly, by (3.32), if $\tau_{\mathbf{X},T} \in (a,T)$ then $\tau_{\mathbf{X},[T,2T]} \in (T+a, 2T)$, and $X\big(\tau_{\mathbf{X},T}\big) > \sup_{t\in[T,T+a]} X(t)$. Therefore,

$$P\big(\tau_{\mathbf{X},T} \in (0, b-a)\big) = P\big(\tau_{\mathbf{X},T} \in (a,b)\big)$$

for any $0 < a < b < T$, which proves the uniformity of the distribution of $\tau_{\mathbf{X},T}$. □

Remark 3.4. A simple special case of the process in Remark 3.2 shows that the statement of part (b) of Theorem 3.3 may fail without Assumption U_T.

We take, for clarity, a specific function x. Let $x(t) = 1 - 2|t|$ for $|t| \le 1/2$ and extend x to a periodic function with period 1. Then for any $T > 1$, the leftmost location of the supremum in the interval $[0,T]$ of the process $(X(t) = x(t+U),\ t \in \mathbb{R})$ is in the interval $(0,1)$ with probability 1, and (as

we already know) this location is not uniformly distributed between 0 and T.

None of the statement of Theorem 3.3 holds, in general, without Assumption L, as the following example shows.

Example 3.5. Let $X(t) = e^{-t/2}B(e^t)$, $t \geq 0$, where $(B(t))$ is the standard Brownian motion. Then $\mathbf{X}$ is a stationary Gaussian process, the Ornstein-Uhlenbeck process. It is, clearly, sample continuous, and the strong Markov property of the Brownian motion shows that, for any $T > 0$, it satisfies Assumption U_T. It is clear that Assumption L fails for the Ornstein-Uhlenbeck process.

By the law of iterated logarithm for the Brownian motion we see that, on a set of probability 1, in any interval $(0, \varepsilon)$ with $\varepsilon > 0$ there is a point t such that $X(t) > X(0)$. Therefore, $P(\tau_{\mathbf{X},T} = 0) = 0$ and, similarly, $P(\tau_{\mathbf{X},T} = T) = 0$ for any $T > 0$.

It is also easy to show, using the basic properties of the Brownian motion, that the density $f_{\mathbf{X},T}$ is not bounded near each of the two endpoints of the interval $[0, T]$, so that both statements of Theorem 3.3 fail for this process.

4. Universal upper bounds on the density

The upper bounds in part (b) of Theorem 3.1 turn out to be the best possible pointwise, as is shown in the following result.

Proposition 4.1. *For each $0 < t < T$ and any number smaller than the upper bound given in* (3.1)*, there is a sample continuous stationary process satisfying Assumption U_T and Assumption L for which the right continuous version of the density $f_{\mathbf{X},T}(t)$ of the supremum location at time t exceeds that number.*

Proof. By symmetry, it is enough to show that for any $0 < t < T$ and any number smaller than $1/t$ there is a stationary process of the required type for which $f_{\mathbf{X},T}(t)$ exceeds that number.

To this end, let $\tau > t$ and let $k \geq 1$ be an integer. We define a periodic function $(x(s),\ s \in \mathbb{R})$ with period $k\tau + 2T$ by defining its values on the

interval $[0, k\tau+2T]$. We set $x(i\tau) = k-i$ for $i = 0, 1, \ldots, k$ and $x(k\tau+2T) = k$. We set, further, for $i = 0, 1, \ldots, k-1$, $x((i+1/2)\tau) = -R$ and also $x\big(k\tau+T\big) = -R$ for a large positive R we describe in a moment. We complete the definition of the function by connecting linearly the values in neighboring points where the function has already been defined. Fix $t < r < \tau$, and choose now R so large that the condition

$$x\big(i\tau\big) > x\big(i\tau - r\big) \tag{4.1}$$

holds for all $i = 1, \ldots, k$. Now define a stationary process by $X(s) = x(s - U)$, $s \in \mathbb{R}$, where U is uniformly distributed between 0 and $k\tau + 2T$. By construction, the process is sample continuous and satisfies Assumption U_T and Assumption L.

If, for $i = 1, \ldots, k$, we have $i\tau - r < U < i\tau$, then the local maximum at $s = i\tau$ of the function $\mathbf{x}$ becomes the global maximum of the process $\mathbf{X}$ over the interval $[0, T]$, and is located in the interval $(0, r)$. This contributes $1/(k\tau + 2T)$ to the value of the density $f_{\mathbf{X},T}$ at each point of the interval $(0, r)$. In particular, since $t \in (0, r)$,

$$f_{\mathbf{X},T}(t) \geq \frac{k}{k\tau + 2T}\,.$$

Since we can take k arbitrarily large, the value of the density can be arbitrarily close to $1/\tau$ and, since τ can be taken arbitrarily close to t, the value of the density can be arbitrarily close to $1/t$. $\square$

Suppose now that the stationary process $\mathbf{X}$ is time reversible, i.e. if $(X(-t),\, t \in \mathbb{R}) \overset{d}{=} (X(t),\, t \in \mathbb{R})$. That would, obviously, be the case for stationary Gaussian processes. If the process satisfies also Assumption U_T, then the distribution of the unique supremum location $\tau_{\mathbf{X},T}$ is symmetric in the interval $[0, T]$, meaning that $\tau_{\mathbf{X},T} \overset{d}{=} T - \tau_{\mathbf{X},T}$. Therefore, the density $f_{\mathbf{X},T}$ satisfies

$$f_{\mathbf{X},T}(t) = f_{\mathbf{X},T}(T - t) \tag{4.2}$$

for all $0 < t < T/2$ that are continuity points of $f_{\mathbf{X},T}$. Even though the upper bound given in part (b) of Theorem 3.1 is symmetric around the middle of the interval $[0, T]$, it turns out that the bounded variation property in part (d)

of Theorem 3.1 provides a better bound in this symmetric case. This bound and its optimality, even within the class of stationary Gaussian processes, is presented in the following result.

Proposition 4.2. *Let $\mathbf{X} = (X(t),\, t \in \mathbb{R})$ be a time reversible stationary sample upper semi-continuous process satisfying Assumption U_T. Then the density $f_{\mathbf{X},T}$ of the unique location of the supremum in the interval $[0,T]$ satisfies*

$$f_{\mathbf{X},T}(t) \leq \begin{cases} \frac{1}{2t} & \text{if } 0 < t \leq \frac{T}{3} \\ \frac{1}{T-t} & \text{if } \frac{T}{3} < t \leq \frac{T}{2} \\ \frac{1}{t} & \text{if } \frac{T}{2} < t \leq \frac{2T}{3} \\ \frac{1}{2(T-t)} & \text{if } \frac{2T}{3} < t < T \end{cases} . \tag{4.3}$$

Furthermore, for each $0 < t < T$ and any number smaller than the upper bound given in (4.3)*, there is a sample continuous Gaussian process satisfying Assumption U_T and Assumption L for which the density $f_{\mathbf{X},T}(t)$ exceeds that number.*

Proof. Since the density $f_{\mathbf{X},T}$ is right continuous, it is enough to consider only continuity points of the density and, by (4.2), it is enough to consider $0 < t < T/2$. Then $T-t$ is also a continuity point of the density. Denote $a = \inf_{0<s\leq t} f_{\mathbf{X},T}(s)$, $b = \inf_{t<s<T/2} f_{\mathbf{X},T}(s)$. Note that, given $\varepsilon > 0$, there is a continuity point of the density $u \in (0,t]$ such that $f_{\mathbf{X},T}(u) \leq a+\varepsilon$, and there is a continuity point of the density $v \in [t, T/2]$ such that $f_{\mathbf{X},T}(v) \leq b+\varepsilon$. Observe also that

$$at + b(T/2 - t) \leq \int_0^{T/2} f_{\mathbf{X},T}(s)\, ds \leq \frac{1}{2} . \tag{4.4}$$

Furthermore, applying the total variation bound (3.3) to the interval $[u, T-u]$ gives us

$$\begin{aligned} 2(a+\varepsilon) &\geq f_{\mathbf{X},T}(u) + f_{\mathbf{X},T}(T-u) \\ &\geq \big|f_{\mathbf{X},T}(t) - f_{\mathbf{X},T}(u)\big| + \big|f_{\mathbf{X},T}(v) - f_{\mathbf{X},T}(t)\big| \\ &+\big|f_{\mathbf{X},T}(T-v) - f_{\mathbf{X},T}(v)\big| + \big|f_{\mathbf{X},T}(T-t) - f_{\mathbf{X},T}(T-v)\big| \\ &+\big|f_{\mathbf{X},T}(T-u) - f_{\mathbf{X},T}(T-t)\big| \\ &\geq 2\big(f_{\mathbf{X},T}(t) - a - \varepsilon\big)_+ + 2\big(f_{\mathbf{X},T}(t) - b - \varepsilon\big)_+ . \end{aligned}$$

Letting $\varepsilon \to 0$ and recalling that $a \le f_{\mathbf{X},T}(t)$ and $b \le f_{\mathbf{X},T}(t)$, we obtain

$$f_{\mathbf{X},T}(t) \le a + b/2\,. \tag{4.5}$$

Since $b \le f_{\mathbf{X},T}(t)$, this implies that

$$b \le 2a\,. \tag{4.6}$$

If $0 < t \le T/3$, then the largest value of the right hand side of (4.5) under the constraint (4.4) requires taking a as large as possible and b as small as possible. Taking $a = 1/2t$ and $b = 0$ in (4.5) results in the upper bound given in (4.3) in this range. If $T/3 < t \le T/2$, then the largest value of the right hand side of (4.5) under the constraint (4.4) requires taking a as small as possible and b as large as possible. By (4.6), we have to take $a = 1/2(T-t)$, $b = 1/(T-t)$ in (4.5), which results in the upper bound given in (4.3) in this case.

It remains to prove the optimality part of the statement of the corollary. By symmetry it is enough to consider $0 < t \le T/2$. Fix such t. Let $\varepsilon > 0$ be a small number and $h > 0$ be a large number, rationally independent of $t + \varepsilon$. Consider a stationary Gaussian process given by

$$X(s) = G_1 \cos\left(\frac{2\pi}{t+\varepsilon}s\right) + G_2 \sin\left(\frac{2\pi}{t+\varepsilon}s\right)$$
$$+G_3 \cos\left(\frac{2\pi}{h}s\right) + G_4 \sin\left(\frac{2\pi}{h}s\right),\ \ s \in \mathbb{R}\,,$$

where $G_1, \ldots, G_4$ are i.i.d. standard normal random variables. The process is, clearly, sample continuous, and it satisfies Assumption L. Furthermore, rational independence of $t+\varepsilon$ and h implies that, on a set of probability 1, the process $\mathbf{X}$ has different values at all of its local maxima, hence Assumption U_T is satisfied for any $T > 0$. Note that we can write

$$X(s) = A_1 \cos\left(\frac{2\pi}{t+\varepsilon}s + U_1\right) + A_2 \cos\left(\frac{2\pi}{h}s + U_2\right) := X_1(s) + X_2(s),\ s \in \mathbb{R}\,,$$

where A_1 and A_2 have the density $xe^{-x^2/2}$ on $(0, \infty)$, and U_1 and U_2 are uniformly distributed between 0 and 2π, with all 4 random variables being independent. Clearly, the leftmost location of the supremum of the process $\mathbf{X}_1$ is at

$$\tau_1 = (t+\varepsilon)\frac{2\pi - U_1}{2\pi},$$

which is uniformly distributed between 0 and $t+\varepsilon$. On the event $E = \{0 < U_2 < \pi - 2\pi T/h\}$ the process $\mathbf{X}_2$ is decreasing on $[0,T]$, so the value of the sum $\mathbf{X}$ at the leftmost supremum of $\mathbf{X}_1$ exceeds the value of the sum at all the other locations of the supremum of $\mathbf{X}_1$ in the interval $[0,T]$. If the supremum of the sum remained at τ_1, the density of that unique supremum would be at least $P(E)/(t+\varepsilon)$ at each point of the interval $(0, t+\varepsilon)$. Since $P(E) \to 1/2$ as $h \to \infty$, the value of the density at t would exceed any value smaller than $1/2t$ after taking h large and ε small. The location of the supremum of the sum does not remain at τ_1 but, instead, moves to $\tau_2 = \tau_2(A_1, A_2, U_1, U_2)$ defined by

$$\tau_2 = \sup\left\{s \le \tau_1 : \frac{A_1}{t+\varepsilon}\sin\left(\frac{2\pi}{t+\varepsilon}s + U_1\right) + \frac{A_2}{h}\sin\left(\frac{2\pi}{h}s + U_2\right) = 0\right\}.$$

For large h, τ_2 is nearly identical to τ_1, and straightforward but somewhat tedious calculus based on the implicit function theorem shows that the above statement remains true for τ_2: the contribution of the event E to the density of the unique supremum of the process $\mathbf{X}$ would exceed any value smaller than $1/2t$ at any point of the interval $(0, t+\varepsilon)$ after taking h large and ε small. We omit the details.

We have shown the optimality of the upper bound given in (4.3) in the case $0 < t \le T/3$. It remains to consider the case $T/3 < t \le T/2$. We will use again a two-wave stationary Gaussian process, but with a slightly different twist. Let $\varepsilon > 0$ be a small number, $h > 0$ a large number and $r > 0$ a fixed number that is rationally independent of $T - t + \varepsilon$. Consider a stationary Gaussian process given by

$$X(s) = A_1\cos\left(\frac{2\pi}{T-t+\varepsilon}s + U_1\right) + \frac{1}{h}A_2\cos\left(\frac{2\pi}{r}s + U_2\right)$$

$$:= X_1(s) + X_2(s),\ s \in \mathbb{R},$$

where A_1, A_2, U_1 and U_2 are as above. As above, $\mathbf{X}$ is a sample continuous Gaussian process satisfying Assumption L and Assumption U_T. Now the leftmost location of the supremum of the process $\mathbf{X}_1$ is at

$$\tau_1 = (T - t + \varepsilon)\frac{2\pi - U_1}{2\pi},$$

which is uniformly distributed between 0 and $T-t+\varepsilon$. Further, if $\tau_1 > t-\varepsilon/2$, then τ_1 is the unique supremum of $\mathbf{X}_1$ in the interval $[0,T]$. If the supremum of the sum $\mathbf{X}$ remained at τ_1, then the density of the supremum location at the point t would be at least $1/(T-t+\varepsilon)$, which would then exceed any value smaller than $1/(T-t)$ after taking ε small. The location of the supremum of $\mathbf{X}$ does not remain at τ_1, but instead moves to the unique for large h point $\tau_2 = \tau_2(A_1, A_2, U_1, U_2)$ in $[0,T]$ satisfying

$$\frac{A_1}{T-t+\varepsilon}\sin\left(\frac{2\pi}{T-t+\varepsilon}\tau_2 + U_1\right) + \frac{A_2}{hr}\sin\left(\frac{2\pi}{r}\tau_2 + U_2\right) = 0\,.$$

For large h, τ_2 is nearly identical to τ_1 and, as above, using the implicit value theorem allows us to conclude that, for any value smaller than $1/(T-t)$, the value of the density of τ_2 in the interval $(t-\varepsilon/2, T-t+\varepsilon)$ exceeds that value after taking ε small and h large. This proves the optimality of the upper bound given in (4.3) in all cases. □

References

R. Adler, O. Bobrowski, M. Borman, E. Subag and S. Weinberger (2010): Persistent homology for random fields and complexes. In *Borrowing Strength: Theory Powering Applications âĂŞ A Festschrift for Lawrence D. Brown*, J. Berger, T. Cai and I. Johnstone, editors. Institute of Mathematical Statistics, Beachwood, Ohio, pp. 124–143.

R. Adler and J. Taylor (2007): *Random Fields and Geometry*. Springer, New York.

M. Leadbetter, G. Lindgren and H. Rootzén (1983): *Extremes and Related Properties of Random Sequences and Processes*. Springer Verlag, New York.

G. Samorodnitsky and Y. Shen (2011): Distribution of the supremum location of stationary processes. Technical report.

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